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On the existence of solutions for P-Laplacian systems with integral Boundary conditions

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ABSTRACT

The boundary value problems arise from many fields of applied mathematics and physics. Various applications of boundary value problems to applied mathematical sciences are well documented in the literature. In this paper, I study the solutions for a class of (p,q) Laplacian system. Using the fixed point theorems in cones the existence of positive solutions is proved. Sufficient conditions are provided under which this system has solution.

Keywords: Positive solution, Integral boundary condition, (p,q) - Laplacian system.

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INTRODUCTION

In this paper, we study the existence of positive solutions for the following (p,q) Laplacian system.

$$\begin{cases}\phi_{p_1}(u'))' + m(t)f(t,u,v) = 0, t \in (0,1)\\ \phi_{p_2}(v'))' + n(t)g(t,u,v) = 0\,,\end{cases} \qquad (1)$$
$$\begin{cases}u(0) - au'(0) = \int_0^1 g_1(s)u(s)ds\\ u(1) + bu'(1) = \int_0^1 g_2(s)u(s)ds\end{cases}$$
$$\begin{cases}v(0) - cv'(0) = \int_0^1 h_1(s)v(s)ds\\ v(1) + dv'(1) = \int_0^1 h_2(s)v(s)ds\end{cases}$$

Where

$$\emptyset_p(s) = |s|^{p-2}s, \qquad p > 1,\ \emptyset_q = \left(\emptyset_p\right)^{-1}, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

We assume that the following conditions:

$H_1)$ $f \in C\big([0,1] \times [0,+\infty) \times R, [0,+\infty)\big); m,n \in C\big([0,1],[0,+\infty)\big);$
$H_2)$ g_i, h_i *are nonnegative* , $\int_0^1 g_i\,(t)dt, \int_0^1 h_i\,(t)dt \in [0,1).$

Recently, many authors have studied the boundary value problems. The existence of solutions for the p-Laplacian equation

$$\begin{cases}(\phi_p(u'))' + h(t)f(t,u,v) = 0, t \in (0,1)\\ \quad u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds,\\ \quad u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds\end{cases}$$

Was studied by (Yang, Wang, 2017).

PRELIMINARIES

Definition (1) (Deimling, 1985) Let $(X, \|.\|)$ be a real Banach space and a non-empty, closed, convex C subset of X is called a Cone of X, If it satisfies the following conditions:
i) If $x \in C$ and $\lambda \geq 0$ implies that $\lambda x \in C$, ii) If $x \in C$ and $-x \in C$ implies that $x = 0$,
Every cone C subset of X includes an ordering in X which is given by $x \leq y$ if and only if $y - x \in C$.

Definition (2). (Deimling, 1985) A map $\psi: P \to [0,+\infty)$ is called nonnegative continuous concave functional provided ψ is nonnegative, continuous and satisfies

$\psi(tx+(1-t)y) \geq t\psi(x)+(1-t)\,\psi(y)$ for all $x, y \in P$ and $t \in [0,1]$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of X $\beta: P \to [0,+\infty)$ is continuous and $\beta(tx+(1-t)y) \geq t\beta(x)+(1-t)\,\beta(y)$ for all $x, y \in P$ and $t \in [0,1]$.

Definition (3). (Kong, 2006) Let γ, θ be a nonnegative, continuous convex functional on P, α be a nonnegative, continuous concave functional on P, ψ be a nonnegative, continuous functional on P. Then for positive real number a, b, c and d, we define the following sets

$P(\gamma, d) = \{x \in P: \gamma(x) \leq \mathrm{d}\}$,
$P(\gamma, \alpha; b, d) = \{x \in P: b \leq \alpha(x), \gamma(x) \leq \mathrm{d}\}$,
$P(\gamma, \theta, \alpha; b, c, d) = \{x \in P: b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq \mathrm{d}\}$,
$R(\gamma, \psi; a, d) = \{x \in P: a \leq \psi(x), \gamma(x) \leq \mathrm{d}\}$,

Theorem (4) (Avery & Peterson, 2001) Let P be a cone in a real Banach space E. Let γ, θ be nonnegative, continuous convex functional on P, α be a nonnegative, continuous concave functional on P and ψ be a nonnegative, continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x), for\ 0 \leq \lambda \leq 1$, such that for some positive number M and d, $\alpha(x) \leq \psi(x)$ and $\|x\| \leq M\gamma(x)$, for $x \in \overline{P(\gamma, d)}$.
Suppose that $T: \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b and c with $a < b$ such that
i) $\{x \in P(\gamma, \theta, \alpha; b, c, d): \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha; b, c, d)$;
ii) $\alpha(Tx) > b$ for $x \in P(\gamma, \alpha; b, d)$ with $\theta(Tx) > c$;
iii) $0 \notin R(\gamma, \psi; a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi; a, d)$ with $\psi(x) = a$.
Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that $\gamma(x_i) \leq d$ for i=1,2,3; $b < \alpha(x_1)$; $a < \psi(x_2)$, with $\alpha(x_2) < b$; and $\psi(x_3) < a$.
Let,
$K = \{(u, v)|u(t) \geq 0, v(t) \geq 0, u, v\ are\ concave\ on\ [0,1]\}$
be a Banach space where

$$\|(u, v)\| = \frac{\|u\| + \|v\|}{2}$$

and

$$\|u\| = max\left\{\max_{0 \leq t \leq 1}|u(t)|, \max_{0 \leq t \leq 1}|u'(t)|\right\}.$$

Lemma (2). Suppose that H_2 hold, then for $w_1, w_2 \in C[0,1]$ and $w_i \geq 0$, the problem

$$\begin{cases} \phi_{p_1}(u'))' + w_1(t) = 0, t \in (0,1) \\ u(0) - au'(0) = \int_0^1 g_1(s)u(s)ds, \\ u(1) + bu'(1) = \int_0^1 g_2(s)u(s)ds \end{cases} \quad (2)$$

$$\begin{cases} \phi_{p_2}(v'))' + w_2(t) = 0, t \in (0,1) \\ v(0) - cv'(0) = \int_0^1 h_1(s)v(s)ds \\ v(1) + dv'(1) = \int_0^1 h_2(s)v(s)ds \end{cases}$$

Has a unique solution of the form

$$u(t) = \frac{-b\phi_{q_1}(H_{w_1}-\int_0^1 w_1(\tau)d\tau-\int_0^1 g_2(s)\int_s^1 \phi_{q_1}(H_{w_1}-\int_0^l w_1(\tau)d\tau)dlds}{1-\int_0^1 g_2(s)ds}$$
$$-\int_t^1 \phi_{q_1}(H_{w_1} - \int_0^s w_1(\tau)d\tau)ds \quad (3)$$

$$v(t) = \frac{-d\phi_{q_2}(H_{w_2}-\int_0^1 w_2(\tau)d\tau-\int_0^1 h_2(s)\int_s^1 \phi_{q_2}(H_{w_2}-\int_0^l w_2(\tau)d\tau)dlds}{1-\int_0^1 h_2(s)ds}$$
$$-\int_t^1 \phi_{q_2}(H_{w_2} - \int_0^s w_2(\tau)d\tau)ds$$

Or

$$u(t) = \frac{a\phi_{q_1}(H_{w_1})+\int_0^1 g_1(s)\int_0^s \phi_{q_1}(H_{w_1}-\int_0^l w_1(\tau)d\tau)dlds}{1-\int_0^1 g_1(s)ds}$$
$$+\int_0^t \phi_{q_1}(H_{w_1} - \int_0^s w_1(\tau)d\tau)ds \quad (4)$$

and $v(t) = \dfrac{c\phi_{q_2}(H_{w_2})+\int_0^1 h_1(s)\int_0^s \phi_{q_2}(H_{w_2}-\int_0^l w_1(\tau)d\tau)dlds}{1-\int_0^1 h_1(s)ds}$

$$+\int_0^t \phi_{q_2}(H_{w_2} - \int_0^s w_1(\tau)d\tau)ds,$$

where H_{w_1} satisfies

$$a\phi_{q_1}(H_{w_1}) = \int_0^1 g_1(s)\int_s^1 \phi_{q_1}(H_{w_1} - \int_0^l w_1(\tau)d\tau)dlds - \int_0^1 \phi_{q_1}(H_{w_1} - \int_0^l w_1(\tau)d\tau)dl$$
$$-\frac{(1-\int_0^1 g_1(s)ds)(b\phi_{q_1}(H_{w_1}-\int_0^1 w_1(\tau)d\tau)+\int_0^1 g_2(s)\int_s^1 \phi_{q_1}(H_{w_1}-\int_0^l w_1(\tau)d\tau)dlds}{1-\int_0^1 g_2(s)ds} \quad (5)$$

and

$$c\phi_{q_2}(H_{w_2}) = \int_0^1 h_1(s)\int_s^1 \phi_{q_2}(H_{w_2} - \int_0^l w_2(\tau)d\tau)dlds - \int_0^1 \phi_{q_2}(H_{w_2} - \int_0^l w_2(\tau)d\tau)dl$$
$$-\frac{(1-\int_0^1 h_1(s)ds)(d\phi_{q_2}(H_{w_2}-\int_0^1 w_2(\tau)d\tau)+\int_0^1 h_2(s)\int_s^1 \phi_{q_2}(H_{w_2}-\int_0^l w_2(\tau)d\tau)dlds}{1-\int_0^1 h_2(s)ds} \quad (6)$$

Proof. It is easy to see that (u(t),v(t)) is the solution of the problem (2).

Lemma (6). (Yang & Wang, 2017) Suppose H_2 hold, for every $w \in C[0,1]$ there exists a unique constant $0 < H_{w_1} < \int_0^l w_1(\tau)d\tau$ satisfying (5) and there is a $0 < \sigma < 1$ such that $H_{w_1} = \int_0^\sigma w_1(\tau)d\tau$.
We define the operator

$$T: K \to K,\ \ T(u, v) = \big(T_1(u, v), T_2(u, v)\big),$$

and,

$$T_1(u, v) = \begin{cases} \dfrac{a\phi_{q_1}(\int_0^\sigma m(\tau)f(\tau,u(\tau),v(\tau))d\tau)+\int_0^1 g_1(s)\int_0^s \phi_{q_1}(\int_v^\sigma m(\tau)f(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 g_1(s)ds} \\ \quad +\int_0^t \phi_{q_1}(\int_s^\sigma m(\tau)f(\tau,u(\tau),v(\tau))d\tau)ds, 0\leq t,s\leq\sigma, \\ \dfrac{b\phi_{q_1}(\int_\sigma^1 m(\tau)f(\tau,u(\tau),v(\tau))d\tau)+\int_0^1 g_2(s)\int_s^1 \phi_{q_1}(\int_\sigma^l m(\tau)f(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 g_2(s)ds} \\ \quad +\int_t^1 \phi_{q_1}(\int_\sigma^s m(\tau)f(\tau,u(\tau),v(\tau))d\tau)ds, \sigma\leq t, \qquad ,\sigma\leq t,s\leq 1. \end{cases} \quad (7)$$

$$T_2(u, v) = \begin{cases} \dfrac{c\phi_{q_2}(\int_0^\sigma n(\tau)g(\tau,u(\tau),v(\tau))d\tau)+\int_0^1 h_1(s)\int_0^s \phi_{q_2}(\int_l^\sigma n(\tau)g(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 h_1(s)ds} \\ \quad +\int_0^t \phi_{q_2}(\int_s^\sigma n(\tau)g(\tau,u(\tau),v(\tau))d\tau)ds, 0\leq t,s\leq\sigma, \\ \dfrac{d\phi_{q_2}(\int_\sigma^1 n(\tau)g(\tau,u(\tau),v(\tau))d\tau)+\int_0^1 h_2(s)\int_s^1 \phi_{q_2}(\int_\sigma^l n(\tau)g(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 h_2(s)ds} \\ \quad +\int_t^1 \phi_{q_2}(\int_\sigma^s n(\tau)g(\tau,u(\tau),v(\tau))d\tau)ds, \sigma\leq t, \qquad ,\sigma\leq t,s\leq 1. \end{cases} \quad (8)$$

So $(u,v)(t)$ is a solution of boundary value problems (1) if and only if $(u,v)(t)$ is a fixed point of the operator T and T is completely continuous.

Lemma (7). For Operator T is defined by (7) , (8) we have, $\max_{0\le t\le 1}\|T(u,v)(t)\| = \|T(u(\sigma),v(\sigma))\|, \forall\, (u,v)\in K$

Proof. We can see

$$(T_1(u,v))'(t) = \phi_{q_1}(\int_t^\sigma m(\tau)f(\tau,u(\tau),v(\tau))d\tau) \ge 0, \forall t\in(0,\sigma]$$

and

$$(T_2(u,v))'(t) = \phi_{q_2}(\int_t^\sigma n(\tau)g(\tau,u(\tau),v(\tau))d\tau) \ge 0, \forall t \in [\sigma,1),$$

then we have

$$\max_{0\le t\le 1}\|T(u,v)(t)\| = \|T(u(\sigma),v(\sigma))\|, \forall\, (u,v)\in K. \blacksquare$$

Lemma (8). (Liu, 2004) Let $u(t)\ge 0$, and U is a concave function, then we have

$$\min_{\sigma\le t\le 1-\sigma} u(t) \ge \sigma \max_{0\le t\le 1} u(t), \forall \sigma\in(0,\frac{1}{2}).$$

Lemma (9). If H_2 holds, then for $(u,v)\in K$, we have

$$\max_{0\le t\le 1} u(t) \le \frac{(1+a)\max_{0\le t\le 1} u'(t)}{1-\int_0^1 g_1(s)ds},$$

$$\max_{0\le t\le 1} v(t) \le \frac{(1+c)\max_{0\le t\le 1} v'(t)}{1-\int_0^1 h_1(s)ds},$$

Proof. It is easy to prove.

RESULTS

Let $M = \frac{1+a}{1-\int_0^1 g_1(s)ds}, M' = \frac{1+c}{1-\int_0^1 h_1(s)ds}, M'' = \min\{M, M'\}$

$L_1 = \phi_{q_1}(\int_0^1 m(\tau)d\tau)$, $L'_1 = \phi_{q_2}(\int_0^1 n(\tau)d\tau)$,

$$N_1 = \frac{a\phi_{q_1}(\int_0^\sigma m(\tau)f(\tau,u(\tau),v(\tau))d\tau) + \int_0^1 g_1(s)\int_0^s \phi_{q_1}(\int_l^\sigma m(\tau)f(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 g_1(s)ds}$$

$$N_2 = \frac{b\phi_{q_1}(\int_\sigma^1 m(\tau)f(\tau,u(\tau),v(\tau))d\tau) + \int_0^1 g_2(s)\int_s^1 \phi_{q_1}(\int_\sigma^l m(\tau)f(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 g_2(s)ds}$$

$$N'_1 = \frac{c\phi_{q_2}(\int_0^\sigma n(\tau)g(\tau,u(\tau),v(\tau))d\tau) + \int_0^1 h_1(s)\int_0^s \phi_{q_2}(\int_l^\sigma n(\tau)g(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 h_1(s)ds}$$

$$N'_2 = \frac{d\phi_{q_2}(\int_\sigma^1 n(\tau)g(\tau,u(\tau),v(\tau))d\tau) + \int_0^1 h_2(s)\int_s^1 \phi_{q_2}(\int_\sigma^l n(\tau)g(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 h_2(s)ds}$$

$$L_2 = min\left\{\int_0^\sigma \phi_{q_1}(\int_s^\sigma m(\tau)d\tau)ds, \phi_{q_1}(\int_\sigma^{1-\sigma} m(\tau)d\tau)\right\},$$

$$L'_2 = min\left\{\int_0^\sigma \phi_{q_2}(\int_s^\sigma n(\tau)d\tau)ds, \phi_{q_2}(\int_\sigma^{1-\sigma} n(\tau)d\tau)\right\},$$

We define $\gamma(u,v) = \max_{0\le t\le 1}\|(u'(t),v'(t))\|$,

$\theta(u,v) = \psi(u,v) = \max_{0\le t\le 1}\|(u(t),v(t))\|$,

$\alpha(u,v) = \min_{\sigma\le t\le 1-\sigma}\|(u(t),v(t))\|, 0<\sigma<\frac{1}{2}$,

So γ,θ are continuous convex and α is continuous concave and ψ is continuous on K.

By Lemma 8 , 9 we have: $\sigma\theta(u,v) \le \alpha(u,v) \le \theta(u,v) = \psi(u,v)$, $\psi(\lambda u,\lambda v) = \lambda\psi(u,v), \forall\lambda\in[0,1]$

Theorem 10. If H_1, H_2 hold and there exist positive real numbers r, s, d, $\sigma\in(0,\frac{1}{2})$ with $0<r<s<\sigma d$ such that

$D_1) f(t,u,v) \le \phi_{p_1}\left(\frac{d}{2L_1}\right), g(t,u,v) \le \phi_{p_2}\left(\frac{d}{2L'_1}\right)$, $(t,u,v)\in[0,1]\times[0,M''d]\times[-d,d]$;

$D_2) f(t,u,v) > \phi_{p_1}\left(\frac{w}{\sigma L_2}\right), g(t,u,v) > \phi_{p_2}\left(\frac{w}{\sigma L'_2}\right)$, $(t,u,v)\in[\sigma,1-\sigma]\times\left[w,\frac{w}{\sigma}\right]\times[-d,d]$;

$D_3) f(t,u,v) < \phi_{p_1}\left(\frac{r(1-\int_0^1 g_2(s)ds}{(b+1)L_1}\right), g(t,u,v) < \phi_{p_2}\left(\frac{r(1-\int_0^1 h_2(s)ds}{(d+1)L'_1}\right)$, $(t,u,v)\in[0,1]\times[0,r]\times[-d,d]$;

Then the boundary value problem (1) has at least three positive solutions

$$(u_1,v_1),(u_2,v_2),(u_3,v_3)\in\overline{P(\gamma,d)}$$

such that $\max_{0\le t\le 1}\|(u'_i(t),v'_i(t))\| \le d$, for i=1,2,3;

$w < \min_{\sigma\le t\le 1-\sigma}\|(u_1(t),v_1(t))\|$; r $<\max_{0\le t\le 1}\|(u_2(t),v_2(t))\|$ with $\min_{\sigma\le t\le 1-\sigma}\|(u_2(t),v_2(t))\| < w$; and $\max_{0\le t\le 1}\|(u_3(t),v_3(t))\| < r$.

Proof. We show $T:\overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$. If $(u,v)\in\overline{P(\gamma,d)}$, then $\gamma(u,v) = \max_{0\le t\le 1}\|(u'(t),v'(t))\| \le d$, from D_1 and lemma 9, we have

$$\max_{0\le t\le 1}\|(u(t),v(t))\| \le M''\gamma(u,v) \le M''d,$$

$$\begin{aligned}\gamma((T_1(u,v),(T_2(u,v)) &= \max_{0\le t\le 1}\|(T'_1(u,v),(T'_2(u,v))\| \\ &\le \max_{0\le t\le 1}(\phi_{q_1}(\int_t^\sigma m(\tau)f(\tau,u(\tau),v(\tau))d\tau) \\ &\quad + \phi_{q_2}(\int_t^\sigma n(\tau)g(\tau,u(\tau),v(\tau))d\tau) \\ &\le \phi_{q_1}(\int_0^1 m(\tau)f(\tau,u(\tau),v(\tau))d\tau) \\ &\quad + \phi_{q_2}(\int_0^1 n(\tau)g(\tau,u(\tau),v(\tau))d\tau) \\ &\le \frac{d}{2L_1}\phi_{q_1}(\int_0^1 m(\tau)d\tau) + \frac{d}{2L'_1}\phi_{q_2}(\int_0^1 n(\tau)d\tau) \le \frac{d}{2}+\frac{d}{2} \\ &= d.\end{aligned}$$

So $T:\overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

Let $(\ddot{u}(t),\ddot{v}(t)) = \left(\frac{w}{\sigma},\frac{w}{\sigma}\right)$,

then $\gamma(\ddot{u}(t),\ddot{v}(t)) = \max_{0\le t\le 1}\|\ddot{u}'(t),\ddot{v}'(t)\| = 0$,

$$\theta(\ddot{u}(t),\ddot{v}(t)) = \max_{0\le t\le 1}\|\ddot{u}(t),\ddot{v}(t)\| = \frac{w}{\sigma},$$

$$\alpha(\ddot{u}(t),\ddot{v}(t)) = \min_{\sigma\le t\le 1-\sigma}\|(\ddot{u}(t),\ddot{v}(t))\| = \frac{w}{\sigma} > w,$$

So $(\ddot{u}(t),\ddot{v}(t))\in P\left(\gamma,\theta,\alpha;w,\frac{w}{\sigma},d\right)$, by D_2, $f(t,u,v) > \phi_{p_1}\left(\frac{w}{\sigma L_2}\right), g(t,u,v) > \phi_{p_2}\left(\frac{w}{\sigma L'_2}\right)$.

i) Now suppose that $\varepsilon \in (0,\sigma)$, by D_2, we have

$$\alpha((T_1(u,v),(T_2(u,v)) = \min_{\sigma\le t\le 1-\sigma}\|((T_1(u,v),(T_2(u,v))\|$$
$$= \big(T(u,v)\big)(1-\sigma)$$
$$\ge N_2 + \int_{1-\sigma}^{1}\phi_{q_1}(\int_{\sigma}^{l} m(\tau)f\big(\tau,u(\tau),v(\tau)\big)d\tau)dl$$
$$\ge \int_{1-\sigma}^{1}\phi_{q_1}(\int_{\sigma}^{1-\sigma} m(\tau)f\big(\tau,u(\tau),v(\tau)\big)d\tau)dl$$
$$> \sigma\frac{w}{\sigma L_2}\phi_{q_1}(\int_{\sigma}^{1-\sigma} m(\tau)d\tau = w.$$

ii) if $\varepsilon \in (1-\sigma,1)$, D_2 include that

$$\alpha((T_1(u,v),(T_2(u,v)) = \min_{\sigma\le t\le 1-\sigma}\|((T_1(u,v),(T_2(u,v))\| =$$
$$\big(T(u,v)\big)(\sigma) \ge N_1 + \int_0^{\sigma}\phi_{q_1}(\int_{\sigma}^{l} m(\tau)f\big(\tau,u(\tau),v(\tau)\big)d\tau)dl$$
$$\ge \int_0^{\sigma}\phi_{q_1}(\int_{\sigma}^{1-\sigma} m(\tau)f\big(\tau,u(\tau),v(\tau)\big)d\tau)dl \quad >$$
$$\sigma\frac{w}{\sigma L_2}\phi_{q_1}(\int_{\sigma}^{1-\sigma} m(\tau)d\tau = w.$$

iii) if $\varepsilon \in [\sigma,1-\sigma]$, lemma 8 and D_2 show that

$$\alpha((T_1(u,v),(T_2(u,v)) = \min_{\sigma\le t\le 1-\sigma}\|((T_1(u,v),(T_2(u,v))\|$$
$$\ge \sigma T(u,v)\,(\sigma) \ge \sigma(N_1$$
$$+ \int_0^{\sigma}\phi_{q_1}(\int_{l}^{\sigma} m(\tau)f\big(\tau,u(\tau),v(\tau)\big)d\tau)dl$$
$$> \sigma\frac{w}{\sigma L_2} = w.$$

Hence the condition

(i) of theorem 4 was proved.

Suppose that $(u,v) \in P(\gamma,\alpha;w,d), \theta(T(u,v)) > \frac{w}{\sigma}$ lemma 8 shows that

$$\alpha((T_1(u,v),\big(T_2(u,v)\big) = \min_{\sigma\le t\le 1-\sigma}\big\|((T_1(u,v),\big(T_2(u,v)\big)\big\|$$
$$\ge \sigma\max_{0\le t\le 1}\|(T_1(u,v),(T_2(u,v))\|$$
$$= \sigma\theta\big(T(u,v)\big) > w$$

So the condition (ii) of theorem 4 is satisfied.

If $(u,v) \in R(\gamma,\psi;r,d)$, $\psi(u,v) = \max_{0\le t\le 1}\|(u(t),v(t))\| \ge r > 0.$

By D_3 we have

$$\psi(T(u,v)) = \max_{0\le t\le 1}\|T(u(t),v(t))\| = \|T(u,v)(\sigma)\| =$$

$$\frac{1}{2}\left(\frac{b\phi_{q_1}(\int_{\sigma}^{1} m(\tau)f(\tau,u(\tau),v(\tau))d\tau)+\int_0^1 g_2(s)\int_s^1\phi_{q_1}(\int_{\sigma}^{l} m(\tau)f(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 g_2(s)ds}\right)$$
$$+\frac{1}{2}\Big(\int_{\sigma}^{1}\phi_{q_1}(\int_{\sigma}^{l} m(\tau)f\big(\tau,u(\tau),v(\tau)\big)d\tau)dt\Big)+$$
$$\frac{1}{2}\left(\frac{d\phi_{q_2}(\int_{\sigma}^{1} n(\tau)g(\tau,u(\tau),v(\tau))d\tau)+\int_0^1 h_2(s)\int_s^1\phi_{q_2}(\int_{\sigma}^{l} n(\tau)g(\tau,u(\tau),v(\tau))d\tau)dlds}{1-\int_0^1 h_2(s)ds}\right)$$
$$+\frac{1}{2}\Big(\int_{\sigma}^{1}\phi_{q_2}(\int_{\sigma}^{l} n(\tau)g\big(\tau,u(\tau),v(\tau)\big)d\tau)dt\Big) \le$$
$$\frac{1}{2}\left(\frac{b\phi_{q_1}(\int_0^1 m(\tau)f(\tau,u(\tau),v(\tau))d\tau)+\int_0^1 g_2(s)ds\phi_{q_1}(\int_0^{l} m(\tau)f(\tau,u(\tau),v(\tau))d\tau)}{1-\int_0^1 g_2(s)ds}\right)$$
$$+\frac{1}{2}\left(\int_0^1\phi_{q_1}(\int_{\sigma}^{l} m(\tau)f\big(\tau,u(\tau),v(\tau)\big)d\tau)dt\right)+$$
$$\frac{1}{2}\left(\frac{d\phi_{q_2}(\int_0^1 n(\tau)g(\tau,u(\tau),v(\tau))d\tau)+\int_0^1 h_2(s)ds\phi_{q_2}(\int_0^{l} n(\tau)g(\tau,u(\tau),v(\tau))d\tau)}{1-\int_0^1 h_2(s)ds}\right)$$
$$+\frac{1}{2}\left(\int_0^1\phi_{q_2}(\int_0^{l} n(\tau)g\big(\tau,u(\tau),v(\tau)\big)d\tau)dt\right)$$
$$\le \frac{(b+1)L_1}{1-\int_0^1 g_2(s)ds}.\frac{r(1-\int_0^1 g_2(s)ds)}{2(b+1)L_1}$$
$$+\frac{(d+1)L'_1}{1-\int_0^1 h_2(s)ds}.\frac{r(1-\int_0^1 h_2(s)ds)}{2(d+1)L'_1}$$
$$= \frac{r}{2}+\frac{r}{2} = r.$$

This implies that the condition iii) of theorem 4 is proved. So the proof is complete.

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