Fundamental Journals
International Journal of Fundamental Physical Sciences (IJFPS)

Original Research Papers
Open Access Journals
ISSN: 2231-8186

IJFPS, Vol 9 , No 4, pp 55-58 Dec, 2019
DOI: 10.14331/ijfps.2019.330132

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https://www.fundamentaljournals.org/index.php/ijfps

Von Neumann Entropy by Logarithmic Method

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Received Dec 2019
Received in revised: Dec 2019
Published: Dec 2019

ABSTRACT

The Von Neumann entropy plays a central role in the quantum information theory and is a concave function and following the property $0 < \sum_{i\in 1} S(\lambda_i \rho_i) - \sum_{i\in 1} \lambda_i S(\rho_i) \leq -\sum_{i\in 1} \lambda_i (\log \lambda_i)$. In this paper, we introduce a new proof for the linearity of Von Neumann entropy in the rate without using the above inequality. Here the Von Neumann entropy is concave; that is, given weights $0 \leq \lambda_i, i \in I,\ \sum_{i\in 1} \lambda_i = 1$ and density matrices $\rho_i \in B_1^+(H)$. Roughly speaking, we will show that in the rate case, the Von Neumann entropy is linear without using Fannes inequality.

Keywords: Entropy, Von Neumann entropy, linearity

https://doi.org/10.14331/ijfps.2019.330132

INTRODUCTION

Johann Von Neumann first presented the extension of classical Gibbs entropy in quantum statistical mechanics in the famous book published in 1932 (Neumann, 2013) and he described the entropy by a density matrix (Jaynes, 1965).

$$S(\rho) = -Tr(\rho\ \ln\rho) \tag{1}$$

Where ρ is, the one-particle reduced density matrix and Tr has the usual meaning of the trace of a matrix, and ln is the natural matrix logarithm.

A density matrix is a matrix that describes the statistical state of a system in quantum mechanics. If we consider the spectral decomposition of ρ as,

$$\rho = \sum_j r_j |r_j\rangle\langle r_j| \tag{2}$$

Then according to the Clausius Statement, the change of entropy of a system obtained by adding the small portions of heat quantity received by the system divided by the absolute temperature during the heat absorption. In addition, entropy, in intuition, is an amount of uncertainty respect to a physical system, which plays a central role in many fields of physics, mathematics and information theory. In the classical systems, the concept of information entropy introduced by Claude Shannon (Shannon, 1948) in 1948, although its origin goes back to Pauli and Von Neumann (von NEUMANN & BEYER, 1955). Von Neumann introduced the entropy of a quantum state. Shannon, Kullback (Kullback, 1968), and Von Neumann entropies are typical information theory tools, designed to quantify the information content and possibly information loss for various classical and quantum systems in a specified state (Garbaczewski, 2005). The Von Neumann entropy is continuous and represented by Fannes inequality in 1973 (Shannon, 1948) and 2004 (von NEUMANN & BEYER, 1955). Of course, the Von Neumann definition is based on Shannon definition. Therefore, new trends will be interesting about entropy. One of them is that what is happen to the Von Neumann entropy in the rate case? It is known that it is linear. In this paper, we will give a new proof to linearity of it. It is a well concept to describe the quantum-mechanical system by a density matrix ρ .

$$S(\rho) = -Tr(\rho \log \rho) \tag{3}$$

Where ρ, is the one-particle reduced density matrix and Tr has the usual meaning of the trace of a matrix, and log is the natural matrix logarithm. If we consider the spectral decomposition of ρ as,

$$\rho = \sum_j r_j |r_j\rangle\langle r_j| \tag{4}$$

Then the Von Neumann entropy of ρ is the Shannon entropy (Shannon, 1948) of the probability distribution corresponding to its spectrum

$$S(\rho) = -Tr(\rho \ln \rho) = -\sum_j r_j \, log(r_j) \tag{5}$$

The Von Neumann entropy plays a central role, not only in physics and in mathematics but also it will be surprisingly in many fields, specially, in quantum information theory(Jaeger, 2007; Nielsen & Chuang, 2000). Here we provide a new simple proof to the Von Neumann entropy based on the logarithms properties and without using Fannes inequality (Audenaert, 2007; Tomamichel, Colbeck, & Renner, 2010). Since the Von Neumann entropy, is nonlinear, then it is interesting linear in rate.

Example 1: Let us consider the following density matrix (after normalization).

$$\eta = \sum_{n=2}^{\infty} \delta(3n)\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) + \delta(3n+1)\rho^{\otimes n} + \delta(3n+2)\sigma^{\otimes n} \tag{6}$$

Where $\delta(n) = \frac{1}{n(\log n)^2}$. Now, $\delta(3n)\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) \leq \eta$. Then, we have

$$-\log \eta \leq -\log\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) - \log \delta(3n) \tag{7}$$

Therefore

$$G\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) \leq S\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) - \log \delta(3n) \tag{8}$$

Where $G(\rho) = -Tr(\rho \, \log \eta)$ and $S(\rho) = -Tr(\rho \, \log \rho)$. On the other hand, it is known that $Tr[\rho(\log \rho - \log \sigma)] \geq 0$. Therefore,

$$S\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) \leq G\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) \tag{9}$$

Like the above relations, we have

$$S\,(\rho^{\otimes n}) \leq G(\rho^{\otimes n}) \leq S\,(\rho^{\otimes n}) - \log \delta(3n+1) \tag{10}$$

and

$$S\,(\sigma^{\otimes n}) \leq G(\sigma^{\otimes n}) \leq S\,(\sigma^{\otimes n}) - \log \delta(3n+2) \tag{11}$$

G is linear and so we can find the following relation

$$S\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) - S\,(\rho^{\otimes n}) - S\,(\sigma^{\otimes n}) + \log \delta(3n+1) + \log \delta(3n+2) \leq 0 \tag{12}$$

and

$$0 \leq S\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) - S\,(\rho^{\otimes n}) - S\,(\sigma^{\otimes n}) - \log \delta(3n) \tag{13}$$

Since, the limits of $\frac{1}{n}\log \delta(3n)$, $\frac{1}{n}\log \delta(3n+1)$ and $\frac{1}{n}\log \delta(3n+2)$ are zero, then we get,

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) S\left(\frac{\rho^{\otimes n}+\sigma^{\otimes n}}{2}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) S(\rho^{\otimes n}) + \lim_{n\to\infty}\left(\frac{1}{n}\right) S(\sigma^{\otimes n}) \tag{14}$$

we can extend it for a fix k as follows

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) S\left(\frac{\rho_1^{\otimes n}+\rho_2^{\otimes n}+\cdots+\rho_k^{\otimes n}}{k}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right)\left[\frac{S(\rho_1^{\otimes n})+S(\rho_2^{\otimes n})+\cdots+S(\rho_k^{\otimes n})}{k}\right] \tag{15}$$

For example, let us consider the density matrix ρ (in simple case) defined as follows

$$\rho = \lambda_1 \overbrace{|010010010\ldots010010}^{n} >< 010010\ldots010010| + \lambda_2 \overbrace{|000111000111\ldots000111}^{n} > < 000111000\ldots0000111| \tag{16}$$

Then we have

$$\lim_{n\to\infty} \frac{1}{n} I_c(\rho,\phi) = \frac{1}{3} \lambda_1 I_c \left(|010010 > < 010010|, \phi\right) + \lambda_2 I_c \left(|000111 > < 000111|\right) \tag{17}$$

Example 2: We consider the density matrix η denoted by

$$\eta^n = \sum \lambda_{i_1,i_2,\dots i_n} \rho_{i_1} \otimes \rho_{i_1} \otimes \dots \otimes \rho_{i_n} \tag{18}$$

where $\rho_{i_j} = \rho_1, \rho_2, \dots \rho_k$ for a fix $k \le \sqrt{n}$. Using the above process, then we can have

$$\lim_{n\to\infty} \frac{1}{n} I_c(\eta^n, \phi) = \lim_{n\to\infty} \frac{1}{n} [(n_1 S(\rho_1) + n_2 S(\rho_2) + \cdots + n_k S(\rho_k)] \tag{19}$$

where $n_1 + n_2 + \cdots + n_k = n$. For example in the case of $k = 2$, we have

$$\lim_{n\to\infty} \frac{1}{n} I_c(\eta^n, \phi) = \lim_{n\to\infty} \frac{1}{n} [(lS(\rho_1) + (n-l)S(\rho_2)] \tag{20}$$

Now, let (ρ^n) be a density matrix on $B(H)$ which can be entangled. If we want to construct new density, matrices from it to compute the maximum in quantum capacity like the following

$$\eta^n = \sum_{i=1}^{k \le \sqrt{n}} \lambda_i U_i \rho\, U_i^{\dagger}, \tag{21}$$

then we have

$$\lim_{n\to\infty} \frac{1}{n} S(\eta^n) = \lim_{n\to\infty} \frac{1}{n} \lambda_i S\left(U_i \rho\, U_i^{\dagger}, \phi\right) = S(\rho, \phi) \tag{22}$$

Of course, it can be extend more than one density matrix as generator.

THE PROOF OF THEOREM: (for $k_n \le 2^{\sqrt{n}}$)

Theorem 1.
Let us consider the set of density matrices, $\rho_1^{(n)}, \rho_2^{(n)}, \dots, \rho_{k_n}^{(n)}$ on Hilbert space with dimension 2^n , where $k_n \le 2^{\sqrt{n}}$. Then we have 2

$$\lim_{n\to\infty} \frac{1}{n} S\left(\sum_{i=1}^{k_n} \lambda_i^{(n)} \rho_i^{(n)}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{k_n} \lambda_i^{(n)} S\left(\rho_i^{(n)}\right) \tag{22}$$

Where S is the Von Neumann entropy and $\sum_{i=1}^{k_n} \lambda_i^{(n)} = 1$, for any n.

Proof: Let us define the density matrix η as follows

$$\eta = \sum_{n=2}^{\infty} \delta(l_{n-1})(\lambda_1^{(n)} \rho_1^{(n)} + \lambda_2^{(n)} \rho_2^{(n)} + \cdots \lambda_{k_n}^{(n)} \rho_{k_n}^{(n)}) + + \delta(l_{n-1}+1)\rho_1^{(n)} + \cdots + \delta(l_{n-1}+k_n)\rho_{k_n}^{(n)} \tag{23}$$

Where $l_{n-1} = k_1 + k_2 + \cdots + k_{n-1} + n - 1$ and $\delta(n) = \frac{1}{n \log_n^2}$.

Let us define $G(\rho) = -Tr(\rho \log \eta)$, for any density matrix ρ. Now, we have

$$\delta(l_{n-1} + i)\rho_i^{(n)} \le \eta \tag{24}$$

Therefore,

$$-\log \eta \le -\log \delta((l_{n-1} + i) - \log \rho_i^{(n)} \tag{25}$$

and we get

$$-\log \eta \le -\log \eta^{(n)} - \log \delta(l_{n-1}) \tag{26}$$

where $\eta^{(n)} = \lambda_1^{(n)} \rho_1^{(n)} + \lambda_2^{(n)} \rho_2^{(n)} + \cdots + \lambda_{k_n}^{(n)} \rho_{k_n}^{(n)}$ then we have,

$$G\left(\rho_i^{(n)}\right) \le S\left(\rho_i^{(n)}\right) - \log \delta(l_{n-1} + i) \quad , \quad (i = 1,2, \dots k_n) \tag{27}$$

and

$$G\left(\eta^{(n)}\right) \le S\left(\eta^{(n)}\right) - \log \delta(l_{n-1}) \tag{28}$$

Using the positivity of relative entropy, we have,

$$G\left(\rho_i^{(n)}\right) \ge S\left(\rho_i^{(n)}\right) \tag{29}$$

and

$$G\left(\eta^{(n)}\right) \ge S\left(\eta^{(n)}\right) \tag{30}$$

Therefore,

$$S\left(\rho_i^{(n)}\right) \le G\left(\rho_i^{(n)}\right) \le S\left(\rho_i^{(n)}\right) - \log \delta(l_{n-1} + i), \quad i = 1,2, \dots, k_n \tag{31}$$

and

$$S\left(\eta^{(n)}\right) \le G\left(\eta^{(n)}\right) \le S\left(\eta^{(n)}\right) - \log \delta(l_{n-1}) \tag{32}$$

But, G is linear and hence

$$S\left(\eta^{(n)}\right) - \sum_{i=1}^{k_n} \lambda_i^{(n)} S\left(\rho_i^{(n)}\right) + \sum_{i=1}^{k_n} \lambda_i^{(n)} \log \delta(l_{n-1} + i) \sum_{i=1}^{k_n} \lambda_i^{(n)} \log \delta(l_{n-1} + i) \le 0 \tag{33}$$

and

$$0 \le S\left(\eta^{(n)}\right) - \sum_{i=1}^{k_n} \lambda_i^{(n)} S\left(\rho_i^{(n)}\right) - \sum_{i=1}^{k_n} \lambda_i^{(n)} \log \delta(l_{n-1}) \tag{34}$$

It is clear that $l_{n-1} = k_1 + k_2 + \cdots + k_{n-1} + n - 1 \le n2^{\sqrt{n}} + n - 1 \le 2^{\sqrt{n+1}}$, therefore

$$\lim_{n\to\infty} \frac{1}{n} \log \delta(l_{n-1}) \leq \lim_{n\to\infty} \frac{1}{n} \log \delta\left(2^{\sqrt{n+1}}\right) \quad (35)$$

Analogously,

$$0 \geq \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{k_n} \lambda_i^{(n)} \log \delta(l_{n-1} + i) \geq \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{k_n} \lambda_i^{(n)} \log \delta(l_{n-1} + k_n) \geq \lim_{n\to\infty} \frac{1}{n} \log \delta(l_{n-1} + k_n) = 0 \quad (36)$$

Corollary: In the simple case for density matrices, $\rho_1^{\otimes n}$, $\rho_2^{\otimes n}$, ... $\rho_{2\sqrt{n}}^{\otimes n}$, we have

$$\lim_{n\to\infty} \left(\frac{1}{n}\right) S\left(\frac{\rho_1^{\otimes n} + \rho_2^{\otimes n} + \ldots + \rho_{2\sqrt{n}}^{\otimes n}}{2^{\sqrt{n}}}\right) = \lim_{n\to\infty} \left(\frac{1}{n}\right)\left(\frac{S(\rho_1^{\otimes n}) + S(\rho_2^{\otimes n}) + \cdots + S\left(\rho_{2\sqrt{n}}^{\otimes n}\right)}{2^{\sqrt{n}}}\right) \quad (37)$$

Now, let us consider the separable density matrix η^n defined by

$$\eta^n = \sum_{i=1}^{2^{\sqrt{n}}} \lambda_i \rho_{i_1} \otimes \rho_{i_1} \otimes \rho_{i_2} \ldots \otimes \rho_{i_n} \quad (38)$$

where $\rho_{i_j} = \rho_1, \rho_2, \ldots$ for the simple case, and $\sum_i \lambda_i = 1$. For Example

$$\eta^{(n)} = \lambda_1 \rho_1^{(n)} + \lambda_2 \rho_2^{(n)} + \cdots + \lambda_{2\sqrt{n}} \rho_{2\sqrt{n}}^{(n)} \quad (39)$$

Where, for example,

$$\rho_1^{(n)} = \underbrace{\rho_1 \otimes \rho_1 \otimes \rho_1 \otimes \rho_1 \otimes \rho_1 \ldots \otimes \rho_1}_{n\ times} \quad (40)$$

$$\rho_1^{(n)} = \rho_1 \otimes \rho_2 \otimes \rho_1 \otimes \rho_2 \ldots \rho_1 \otimes \rho_2 \quad (41)$$

$$\rho_{2\sqrt{n}}^{(n)} = \rho_2 \otimes \rho_2 \otimes \rho_2 \otimes \rho_2 \ldots \rho_2 \otimes \rho_2 \quad (42)$$

Using the above process we can get

$$\lim_{n\to\infty} \frac{1}{n} I_c(\eta^{(n)}, \phi) = \lim_{n\to\infty} \frac{1}{n 2^{\sqrt{n}}} \left(n_1 I_c(\rho_1, \phi) + n_2 I_c(\rho_2, \phi)\right) \quad (43)$$

where $n_1 + n_2 = n 2^{\sqrt{n}}$ and ϕ is quantum channel and I_c is coherent information.

CONCLUSION

Here we introduced a new proof for the linearity of Von Neumann entropy in the rate without using inequality. Since the Von Neumann entropy, is concave and is, given weights $0 \leq \lambda_i, i \in I$, $\sum_{i\in 1} \lambda_i = 1$ and density matrices $\rho_i \in B_1^+(H)$ therefor in the rate case, the Von Neumann entropy is linear without using Fannes inequality.

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